

# A SHORT PROOF OF A SYMMETRY IDENTITY FOR THE ( $q, \mu, \nu$ )-DEFORMED BINOMIAL DISTRIBUTION

GUILLAUME BARRAQUAND\*

## Abstract

We give a short and elementary proof of a  $(q, \mu, \nu)$ -deformed Binomial distribution identity arising in the study of the  $(q, \mu, \nu)$ -Boson process and the  $(q, \mu, \nu)$ -TASEP. This identity found by Corwin in [4] was a key technical step to prove an intertwining relation between the Markov transition matrices of these two classes of discrete-time Markov chains. This was used in turn to derive exact formulas for a large class of observables of both these processes.

## INTRODUCTION

Zero-range process and exclusion processes are generic stochastic models for transport phenomena on a lattice. Integrability of these models is an important question. In a short letter [5], Evans-Majumdar-Zia considered spatially homogeneous discrete time zero-range processes on periodic domains. They addressed and solved the question of characterizing the jump distributions for which invariant measures are product measures. Povolotsky [6] further examined the precise form of jump distributions allowing solvability by Bethe ansatz, and found the  $(q, \mu, \nu)$ -Boson process and the  $(q, \mu, \nu)$ -TASEP. He also conjectured exact formulas for the model on the infinite lattice. Using a Markov duality between the  $(q, \mu, \nu)$ -Boson process and the  $(q, \mu, \nu)$ -TASEP, Corwin [4] showed a variant of these formulas and provided a method to compute a large class of observables. This can be seen as a generalization of a similar work on  $q$ -TASEP and  $q$ -Boson process performed in [3, 2]. In his proof, the intertwining relation between the two Markov transition matrices essentially boils down to a  $(q, \mu, \nu)$ -deformed Binomial distribution identity [4, Proposition 1.2]. The proof was adapted from [2, Lemma 3.7] which is the  $\nu = 0$  case, and required the use of Heine's summation formula for the basic hypergeometric series  ${}_2\phi_1$ . In the following, we give a short proof of this identity.

## A SYMMETRY PROPERTY FOR THE $(q, \mu, \nu)$ -DEFORMED BINOMIAL DISTRIBUTION

First, we define the three parameter deformation of the Binomial distribution introduced in [6].

**Definition 1.** For  $|q| < 1$ ,  $0 \leq \nu \leq \mu < 1$  and integers  $0 \leq j \leq m$ , define the function

$$\varphi_{q,\mu,\nu}(j|m) = \mu^j \frac{(\nu/\mu; q)_j (\mu; q)_{m-j}}{(\nu; q)_m} \binom{m}{j}_q,$$

where

$$\binom{m}{j}_q = \frac{(q; q)_m}{(q; q)_j (q; q)_{m-j}}$$

are  $q$ -Binomial coefficients with, as usual,

$$(z; q)_n = \prod_{i=0}^{n-1} (1 - q^i z).$$

---

\*Laboratoire de Probabilités et Modèles Aléatoires, Université Paris Diderot, 5 rue Thomas Mann, 75013 PARIS.  
E-mail: barraquand@math.univ-paris-diderot.fr

It happens that for each  $m \in \mathbb{N} \cup \infty$ , this defines a probability distribution on  $\{0, \dots, m\}$ .

**Lemma 1** (Lemma 1.1, [4]). *For any  $|q| < 1$  and  $0 \leq \nu \leq \mu < 1$ ,*

$$\sum_{j=0}^m \varphi_{q,\mu,\nu}(j|m) = 1.$$

*Proof.* As shown in [4], this equation is equivalent to a specialization of some known summation formula for basic hypergeometric series  ${}_2\phi_1$  (Heine's  $q$ -generalizations of Gauss' summation formula).  $\square$

This probability distribution can be seen as a  $q$ -analogue of the Binomial distribution, depending on two parameters  $0 \leq \nu \leq \mu < 1$  and we call it the  $(q, \mu, \nu)$ -Binomial distribution. In [6], various interesting degenerations are studied. We now state and prove the main identity.

**Proposition 1** (Proposition 1.2, [4]). *Let  $X$  (resp.  $Y$ ) be a random variable following the  $(q, \mu, \nu)$ -Binomial distribution on  $\{0, \dots, x\}$  (resp.  $\{0, \dots, y\}$ ). We have*

$$\mathbb{E}[q^{xY}] = \mathbb{E}[q^{yX}].$$

*Proof.* Let  $S_{x,y} := \sum_{j=0}^x \varphi_{q,\mu,\nu}(j|x)q^{jy}$ . We have to show that  $S_{x,y} = S_{y,x}$  for all integers  $x, y \geq 0$ . Our proof is based on the fact that  $S_{x,y}$  satisfies a recurrence relation which is invariant when exchanging the roles of  $x$  and  $y$ . First notice that by lemma 1,  $S_{x,0} = 1$  for all  $x \geq 0$ , and by definition  $S_{0,y} = 1$  for all  $y \geq 0$ .

The Pascal identity for  $q$ -Binomial coefficients, (see 10.0.3 in [1]),

$$\binom{x+1}{j}_q = \binom{x}{j}_q q^j + \binom{x}{j-1}_q,$$

yields

$$\begin{aligned} S_{x+1,y} &= \sum_{j=0}^{x+1} \mu^j \frac{(\nu/\mu; q)_j (\mu; q)_{x+1-j}}{(\nu; q)_{x+1}} \binom{x}{j}_q q^j q^{jy} + \sum_{j=0}^{x+1} \mu^j \frac{(\nu/\mu; q)_j (\mu; q)_{x+1-j}}{(\nu; q)_{x+1}} \binom{x}{j-1}_q q^{jy}, \\ &= \sum_{j=0}^x \varphi_{q,\mu,\nu}(j|x) \frac{1 - \mu q^{x-j}}{1 - \nu q^x} q^j q^{jy} + \sum_{j=0}^x \varphi_{q,\mu,\nu}(j|x) \mu \frac{1 - \nu/\mu q^j}{1 - \nu q^x} q^y q^{jy}. \end{aligned}$$

The last equation can be rewritten

$$\begin{aligned} (1 - \nu q^x) S_{x+1,y} &= (S_{x,y+1} - \mu q^x S_{x,y}) + (\mu q^y (S_{x,y} - \nu/\mu S_{x,y+1})), \\ &= (1 - \nu q^y) S_{x,y+1} + \mu (q^y - q^x) S_{x,y}. \end{aligned}$$

Thus, the sequence  $(S_{x,y})_{(x,y) \in \mathbb{N}^2}$  is completely determined by

$$\begin{cases} (1 - \nu q^x) S_{x+1,y} = (1 - \nu q^y) S_{x,y+1} + \mu (q^y - q^x) S_{x,y}, \\ S_{x,0} = S_{0,y} = 1. \end{cases} \quad (1)$$

Setting  $T_{x,y} = S_{y,x}$ , one notices that the sequence  $(T_{x,y})_{(x,y) \in \mathbb{N}^2}$  enjoys the same recurrence, which concludes the proof.  $\square$

**Remark.** To completely avoid the use of basic hypergeometric series, one would also need a similar proof of lemma 1. One can prove the result by recurrence on  $m$  (as in the proof of [2, lemma 1.3]), but the calculations are less elegant when  $\nu \neq 0$ .

More precisely, fix some  $m$  and suppose that for any  $0 \leq \nu \leq \mu < 1$ ,  $S_{m,0}(q, \mu, \nu) := \sum_{j=0}^m \varphi_{q,\mu,\nu}(j|m) = 1$ . Pascal's identity yields

$$\begin{aligned} S_{m+1,0}(q, \mu, \nu) &= \frac{1-\mu}{1-\nu} S_{m,0}(q, q\mu, q\nu) + \sum_{j=0}^m \varphi_{q,\mu,\nu}(j|m) \mu \frac{1-\nu/\mu q^j}{1-\nu q^m}, \\ &= \frac{1-\mu}{1-\nu} S_{m,0}(q, q\mu, q\nu) + \frac{\mu}{1-\nu q^m} (S_{m,0}(q, \mu, \nu) - \nu/\mu S_{m,1}(q, \mu, \nu)). \end{aligned}$$

Then, using the recurrence formula (1) for  $S_{m,1}(q, \mu, \nu)$ , and applying the recurrence hypothesis, one obtains  $S_{m+1,0}(q, \mu, \nu) = 1$ .

#### ACKNOWLEDGEMENTS

The author would like to thank his advisor Sandrine Péché for her support.

#### REFERENCES

- [1] George E Andrews, Richard Askey, and Ranjan Roy, *Special functions*, vol. 71, Cambridge University Press, 2001.
- [2] Alexei Borodin and Ivan Corwin, *Discrete time  $q$ -taseps*, International Mathematics Research Notices (2013), rnt206.
- [3] Alexei Borodin, Ivan Corwin, and Tomohiro Sasamoto, *From duality to determinants for  $q$ -tasep and aseps*, Ann. Probab., to appear, arXiv:1207.5035 (2012).
- [4] Ivan Corwin, *The  $(q, \mu, \nu)$ -boson process and  $(q, \mu, \nu)$ -tasep*, arXiv preprint arXiv:1401.3321 (2014).
- [5] Martin R Evans, Satya N Majumdar, and Royce KP Zia, *Factorized steady states in mass transport models*, Journal of Physics A: Mathematical and General **37** (2004), no. 25, L275.
- [6] AM Povolotsky, *On the integrability of zero-range chipping models with factorized steady states*, Journal of Physics A: Mathematical and Theoretical **46** (2013), no. 46, 465205.